

The hidden symmetry algebras of a class of quasi-exactly solvable multi dimensional operators

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Abstract

Let $P(N, V)$ denote the vector space of polynomials of maximal degree less than or equal to N in V independent variables. This space is preserved by the enveloping algebra generated by a set of linear, differential operators representing the Lie algebra $gl(V + 1)$. We establish the counterpart of this property for the vector space $P(M, V) \oplus P(N, V)$ for any values of the integers M, N, V . We show that the operators preserving $P(M, V) \oplus P(N, V)$ generate an abstract superalgebra (non linear if $\Delta = |M - N| \geq 2$). A family of algebras is also constructed, extending this particular algebra by $\Delta - 1$ arbitrary complex parameters.

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1 Introduction

Quasi-exactly solvable (QES) equations refer to a class of spectral differential equations for which a part of the spectrum can be obtained by solving algebraic equations [1, 2, 3]. Linear differential operators preserving a finite dimensional space of smooth functions constitute in this respect a basic ingredient in the topic of QES equations.

In the case of operators of one real variable acting on a scalar function, the possibilities of finite dimensional invariant vector spaces are rather limited [4]. Up to a change of the variable and a redefinition of the function, the vector space can only be the set $P(N)$ of polynomials with degree less than or equal to N . The relevant operators are the elements of the enveloping algebra of $sl(2)$ whose generators are suitably represented by three differential operators [1, 4]. The QES equations they define therefore possess an $sl(2)$ hidden symmetry.

When the number of variables or (and) the number of components of the function is (are) larger than one, the number of possible invariant vector spaces and hidden symmetry algebras increases considerably. The scalar QES operators in two variables were classified in [5, 6]. Seven inequivalent spaces of functions appear to be possible. Correspondingly, the hidden symmetry algebras can be of several types, e.g. $sl(2)$, $sl(2) \otimes sl(2)$, $sl(3)$. A few cases of this classification generalise easily to operators involving an arbitrary number of variables. In particular, the case labelled 2.3 in ref. [6] can be extended to the space $P(N, V)$ of polynomials of maximal degree less than or equal to N in their V independent variables. The related algebra is $sl(V + 1)$.

The construction of the matrix operators in one variable preserving the direct sum $P(N_1) \oplus \dots \oplus P(N_k)$ has also been considered [7, 8]. These operators are closely related to graded algebras. As an example, the case $P(N) \oplus P(N + 1)$ is related to the graded Lie algebra $osp(2, 2)$ [9].

The purpose of this paper is to classify the operators preserving the vector

space

$$P(M, V)$$

$\oplus P(N, V)$ for arbitrary values of the integers M, N, V and to construct a series of associative algebras corresponding to the hidden symmetries of these operators. In Sect. 2 we fix the notations and point out the relevant representations of the algebra $gl(n)$. The 2×2 matrix operators preserving the space $P(N, V) \oplus P(M, V)$ are constructed in Sect. 3 and are shown to obey a set of normal ordering rules. In Sect. 4, these ordering rules are modified into sets of commutation and anticommutation relations which fulfil all Jacobi identities. We obtain in this way a series of associative abstract algebras which appear to be labelled by V , by $\Delta = |N - M|$ and by $\Delta - 1$ arbitrary complex parameters. The technical details related to the proof of our main result are given in Sect. 5.

2 Operators preserving $P(N, V)$

Let N, V be two positive integers. Let x_i ($i = 1, \dots, V$) represent V independent real variables. We define the finite dimensional vector space $P(N, V)$ of polynomials in the variables x_i and of maximal total degree N

$$P(N, V) = \text{span} \{x_1^{n_1}, x_2^{n_2} \dots x_V^{n_V}\} \quad , \quad 0 \leq \sum_{j=1}^V n_j \leq N \quad (1)$$

$$P(N, 1) \equiv P(N) = \text{span} \{1, x, \dots, x^N\} . \quad (2)$$

The dimension of $P(N, V)$ is given by

$$1 + V + \frac{V(V+1)}{2} + \dots + \frac{V(V+1) \dots (V+N-1)}{N!} = C_V^{N+V} . \quad (3)$$

The set of linear differential operators preserving $P(N, V)$ can be perceived as the enveloping algebra generated by the following operators

$$\begin{aligned} J_0^0(N) &= D - N \quad , \quad D \equiv \sum_{j=1}^V x_j \frac{\partial}{\partial x_j} \\ J_0^k(N) &= \frac{\partial}{\partial x_k} \quad , \quad k = 1, \dots, V \end{aligned}$$

$$\begin{aligned}
J_k^0(N) &= -x_k(D - N) \quad , \quad k = 1, \dots, V \\
J_k^l(N) &= -x_k \frac{\partial}{\partial x_l} \quad , \quad k, l = 1, \dots, V .
\end{aligned} \tag{4}$$

These $(V + 1)^2$ independent operators fulfil the commutation rules of the Lie algebra $gl(V + 1)$. Acting on the finite dimensional space $P(N, V)$, they lead to an irreducible representation of this algebra. The commutation relations are

$$[J_a^b, J_c^d] = \delta_a^d J_c^b - \delta_c^b J_a^d \quad , \quad a, b, c, d = 0, 1, \dots, V . \tag{5}$$

Within the representation (4), the Casimir operators of $gl(V + 1)$

$$C_p \equiv \sum_{a_1, \dots, a_p=0}^V J_{a_2}^{a_1} J_{a_3}^{a_2} \dots J_{a_1}^{a_p} \quad , \quad p = 1, \dots, V + 1 \tag{6}$$

have the values $C_p = (-1)^p N(N + V)^{p-1}$. The operators J_a^b defined by

$$J_a^b = J_a^b + C \delta_a^b \quad , \tag{7}$$

where C is any operator which commutes with all J 's, satisfy also the relations (5). For instance, this is the case for the $(V + 1)^2 - 1$ independent operators

$$\tilde{J}_a^b = J_a^b - \frac{1}{V + 1} C_1 \delta_a^b \tag{8}$$

(since $\tilde{C}_1 \equiv 0$) which form an irreducible representation of $sl(V + 1)$ when acting on $P(N, V)$. The usual form [1] of the of $sl(2)$ generators

$$\begin{aligned}
J_+(N) &= -\tilde{J}_1^0 = x(x\partial_x - N) \\
J_0(N) &= -\tilde{J}_1^1 = (x\partial_x - \frac{N}{2}) \\
J_-(N) &= \tilde{J}_0^1 = \partial_x
\end{aligned} \tag{9}$$

is recovered for $V = 1$. These operators play a major role in the topic of quasi-exactly solvable equations.

More generally, an element, say A , of the enveloping algebra constructed over the J_a^b (or the \tilde{J}_a^b) is a quasi-exactly solvable operator preserving $P(N, V)$. That is to say that the spectral equation

$$Ap = \lambda p \quad , \quad p \in P(N, V) \tag{10}$$

admits C_V^{V+N} solutions. Recently, the Calogero and Sutherland quantum hamiltonians were shown to be expressible in terms of the operators J_a^b [10], this result reveals the hidden symmetries of these models.

3 Operators preserving $\mathbf{P(M,V)} \oplus \mathbf{P(N,V)}$

We now put the emphasis on the 2×2 matrix operators which preserve the vector space

$$P(M, V) \oplus P(N, V) \quad , \quad \Delta \equiv N - M \quad . \quad (11)$$

Without loss of generality, we assume the integer Δ to be non negative. In order to classify the operators preserving (11) we define a list of generators. First the "diagonal" generators that we choose as

$$J_a^b(N, \Delta) = \begin{pmatrix} J_a^b(N - \Delta) & 0 \\ 0 & J_a^b(N) \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 + \Delta & 0 \\ 0 & 1 - \Delta \end{pmatrix} \delta_a^b \quad (12)$$

for $0 \leq a, b \leq V$. They are built as a direct sum of two operators of the type (4)-(7), translated by (7) in such a way that that $J_0^0(N, \Delta)$ is proportional to the unit matrix. The interest for this translation will appear later.

The "non diagonal" generators naturally split into " Q operators", proportional to the matrix σ_- (as usual $\sigma_{\pm} = (\sigma_1 \pm i\sigma_2)/2$). and " \overline{Q} operators" proportional to the matrix σ_+ . It is convenient to write them by using a multi index $[A] \equiv a_1, a_2, \dots, a_{\Delta}$. For later convenience we also define $[\hat{A}_i]$ as the set $[A]$ where the index a_i has been removed. We choose the non diagonal generators respectively as

$$Q_{[A]} = (-1)^{\delta} x_{a_1} \dots x_{a_{\Delta}} \sigma_- \quad , \quad 0 \leq a_i \leq V \quad , \quad x_0 \equiv 1 \quad (13)$$

where δ represents the overall degree in x_1, \dots, x_V of the monomial $x_{a_1} \dots x_{a_{\Delta}}$ and

$$\overline{Q}^{[B]} = \overline{q}^{[B]} \sigma_+ \quad , \quad 0 \leq b_i \leq V \quad (14)$$

where the scalar operators $\bar{q}^{[B]}$, fully symmetric in their Δ indices b_k , are defined by

$$\begin{aligned}
\bar{q}^{[B]} &= \partial_{b_1} \dots \partial_{b_\Delta} && \text{if } 0 < b_1 \leq b_2 \leq b_3 \dots \leq b_\Delta \\
&(D - N + \Delta - 1) \partial_{b_2} \dots \partial_{b_\Delta} && \text{if } 0 = b_1 < b_2 \leq b_3 \dots \leq b_\Delta \\
&(D - N + \Delta - 1)(D - N + \Delta - 2) \partial_{b_3} \dots \partial_{b_\Delta} && \text{if } 0 = b_1 = b_2 < b_3 \dots \leq b_\Delta \\
&(D - N + \Delta - 1)(D - N + \Delta - 2) \dots (D - N) && \text{if } 0 = b_1 = b_2 = b_3 = b_\Delta .
\end{aligned} \tag{15}$$

The operators $Q_{[A]}$ (and similarly the $\bar{Q}^{[A]}$) are fully symmetric in their Δ indices a_k . Hence there are $C_\Delta^{V+\Delta}$ independent operators of both types. We then have the following proposition.

Proposition 1

The operators preserving the space $P(N - \Delta, V) \oplus P(N, V)$ are the elements of the enveloping algebra constructed over the generators (12),(13),(14) .

This result (whose demonstration follows the same lines as in the scalar case [4]) allows to write formally all the operators preserving (11). However, in order to classify these operators, it is useful to set up normal ordering rules between the generators. In particular, these rules allow to write any product of operators (the enveloping algebra) in a canonical form, e.g. as a sum of terms where, in each term, the Q operators (if any) are written on the left, the \bar{Q} operators (if any) on the right and the J operators in between. As we show next, such rules exist for the operators (12),(13),(14).

Normal ordering rules

The operators (12) obey the commutation rules (5) and assemble into a reducible representation of $gl(V + 1)$ when acting on the vector space (11). The

dimension of it is

$$C_V^{N+V} + C_V^{N+V-\Delta} . \quad (16)$$

By construction, the operators $Q_{[A]}$ (resp. $\overline{Q}^{[A]}$) transform as an irreducible multiplet of dimension $C_\Delta^{V+\Delta}$ under the adjoint action of the generators $J_a^b(N, \Delta)$. More precisely, we have

$$[J_a^b, Q_{[A]}] = k \delta_a^b Q_{[A]} - \sum_{k=1}^{\Delta} \delta_{a_k}^b Q_{[\hat{A}_k, a]} \quad (17)$$

$$[J_a^b, \overline{Q}^{[A]}] = -k \delta_a^b \overline{Q}^{[A]} + \sum_{k=1}^{\Delta} \delta_a^{a_k} \overline{Q}^{[\hat{A}_k, b]} . \quad (18)$$

The explicit form of the generators leads to the value $k = \Delta$. The first Casimir constructed with (12), i.e.

$$T \equiv \sum_{a=0}^V J_a^a(N, \Delta) , \quad (19)$$

plays the role of a grading operator :

$$[T, J_a^b] = 0 \quad , \quad [T, Q_{[A]}] = \Delta V Q_{[A]} \quad , \quad [T, \overline{Q}^{[A]}] = -\Delta V \overline{Q}^{[A]} . \quad (20)$$

The product of any two operators Q (and separately of two \overline{Q} 's) vanishes, hence also their anticommutator

$$\{Q_{[A]}, Q_{[C]}\} = 0 \quad , \quad \{\overline{Q}^{[B]}, \overline{Q}^{[D]}\} = 0 . \quad (21)$$

The evaluation of the anti-commutator $\{Q, \overline{Q}\}$ is more involved. Its form can be guessed from the covariance under $gl(V+1)$, from the symmetries of Q and \overline{Q} in their indices and from the fact that the anti-commutator involves at most derivatives of the order Δ . It is therefore likely that the anticommutator $\{Q, \overline{Q}\}$ should be expressed as a combination of the tensors

$$W_{[A]}^{[B]}(k) \equiv \frac{1}{(\Delta!)^2} S[A] S[B] (J_{a_1}^{b_1} J_{a_2}^{b_2} \dots J_{a_k}^{b_k} \delta_{a_{k+1}}^{b_{k+1}} \dots \delta_{a_\Delta}^{b_\Delta}) \quad (22)$$

where the operator $S[.]$ denotes the sum over all permutations of all indices entering in the argument $[.]$. After calculation, we found the following relations

between (12),(13),(14),

$$\{Q_{[A]}, \overline{Q}^{[B]}\} = \sum_{k=0}^{\Delta} \alpha_k W_{[A]}^{[B]}(k) \quad (23)$$

and the parameters α_k are numbers which are uniquely determined by the polynomial equation

$$\prod_{j=0}^{\Delta-1} (y + j) = \sum_{k=0}^{\Delta} \alpha_k (y + \frac{\Delta-1}{2})^k. \quad (24)$$

As a consequence of (24), the right hand side of (23) is an even (resp. odd) polynomial in the operators J if Δ is even (resp. odd). We would like to stress that this particularly simple expression is due to the labelling of the generators and to the translation used in (12). A priori, the undetermined coefficients α_k could be 2×2 diagonal matrices.

The non vanishing parameters α_k appear only for $k = \Delta, \Delta - 2, \Delta - 4, \dots$ and read as follows for the first few values of Δ :

$$\begin{aligned} \Delta = 1 & \ , \quad \alpha_k : 1 \\ \Delta = 2 & \ , \quad \alpha_k : 1, -\frac{1}{4} \\ \Delta = 3 & \ , \quad \alpha_k : 1, -1 \\ \Delta = 4 & \ , \quad \alpha_k : 1, -\frac{5}{2}, \frac{9}{16} \\ \Delta = 5 & \ , \quad \alpha_k : 1, -5, 4 \\ \Delta = 6 & \ , \quad \alpha_k : 1, -\frac{35}{4}, \frac{259}{16}, -\frac{225}{64}. \end{aligned} \quad (25)$$

4 Abstract algebras

We now investigate the possibility that the operators (12),(13),(14) represent the generators of an abstract associative algebra. We will see that there are two types of such algebras that we note generically $\mathcal{A}(V, \Delta)$ and $\mathcal{B}(V, \Delta)$. A few cases are known to coincide with Lie algebras [9, 7]

$$\mathcal{A}(1, 0) \simeq sl(2) \otimes sl(2) \quad (26)$$

$$\mathcal{A}(1, 1) \simeq osp(2, 2) \simeq spl(2, 1) . \quad (27)$$

For $\Delta > 1$, $\mathcal{A}(1, \Delta)$ corresponds to a non linear superalgebra [7]. The algebra $\mathcal{A}(1, 2)$ was treated in great detail in [11]. Here we want to move away from the case $V = 1$ in order to access the hidden symmetries of the operators preserving (11) in general.

With the aim to promote the normal ordering rules of the previous section into a set of relations defining an abstract associative algebra, we first note that the operators J_a^b (resp. Q, \overline{Q}) should naturally be interpreted as the bosonic (resp. fermionic) generators of the algebra (this refers of course to the most natural choice of the commutator or of the anti-commutator used to exchange the order between these generators). Therefore, we expect some graded algebras to come out. However, it is well known that the knowledge of a particular representation (here (12),(13),(14)) is not sufficient in general to infer the whole algebraic structure : the Jacobi identities are not automatically fulfilled. In the present case, the identities which are not obeyed are those involving a $\{Q, Q\}$ (or a $\{\overline{Q}, \overline{Q}\}$) anticommutator (remember that they vanish). Although we can try to modify the whole set of (anti) commutation relations, we limit our research of the underlying abstract algebras in relaxing only the relation (21). In order to present the way to modify it, a few notations are worth introducing.

Due to its symmetry in the indices $[A]$, the representation defined by the $Q_{[A]}$ (and similarly by the $\overline{Q}^{[A]}$) corresponds to a Young diagram with one line of Δ boxes. The products

$$Q_{a_1 \dots a_\Delta} Q_{c_1 \dots c_\Delta} \quad (28)$$

assemble into a representation of $gl(V + 1)$ under the adjoint action of the operators J . This representation can be decomposed into irreducible pieces. The symmetry of Q is such that the irreducible representations appearing in the decomposition of (28) correspond to the Young diagrams consisting of two lines

with total number of 2Δ boxes. When applied to the anticommutators

$$Q_{a_1 \dots a_\Delta} Q_{c_1 \dots c_\Delta} + Q_{c_1 \dots c_\Delta} Q_{a_1 \dots a_\Delta} , \quad (29)$$

the same decomposition selects only the representations which are symmetric under the exchange $[A] \leftrightarrow [C]$. In terms of Young diagrams they correspond to the diagrams with two lines and total number of 2Δ boxes; the upper line is of length $2\Delta - 2p$ (with $2p \leq \Delta$) and the lower line is of even length $2p$. One Young tableau, corresponding to this Young diagram with fixed p , is obtained by filling the first (resp. the second) line with

$$[a_1, a_2, \dots, a_\Delta, c_{2p+1}, c_{2p+2}, \dots, c_\Delta] \quad , \quad (\text{ resp. } [c_1, c_2, \dots, c_{2p}]) . \quad (30)$$

The Young element S_Y corresponding to this Young tableau reads

$$S_Y = S[a_1, \dots, a_\Delta, c_{2p+1}, \dots, c_\Delta] S[c_1, \dots, c_{2p}] E_x \quad , \quad (31)$$

with

$$E_x \equiv \prod_{k=1}^{2p} (E - (a_k, c_k)) \quad (32)$$

where the operator $S[.]$ (defined previously) denotes the sum over all permutations of all indices appearing in the argument $[.]$, where (a, b) denotes the transposition $a \leftrightarrow b$ and where E is the identity operator.

With these notations, we are ready to describe the conditions we choose for the anticommutations of two Q operators . We restrict them by imposing

$$S_Y \{Q_{[A]}, Q_{[C]}\} = 0 . \quad (33)$$

This corresponds to the vanishing of a particular representation contained in decomposition of the symmetrized product of $Q_{[A]}$ with $Q_{[C]}$ into irreducible representations of $gl(V+1)$. The absent representation is exactly the one related to the Young diagram defined above, characterized by p and by the Young element (31).

We have studied the associativity conditions, i.e. the Jacobi identities, compatible with the relation (5), the relations (17),(18) for an arbitrary value of the parameter k , the relation (23) for arbitrary values of the parameters α_k and the relation (33) for an arbitrary integer p (in fact $2p \leq \Delta$). The results of our calculation is summarized by the following proposition :

Proposition 2

The set of relations (5),(17),(18),(23),(33) are compatible with all the Jacobi identities in two cases only :

1. $p = 0$ and $k = \Delta$
2. Δ even, $\Delta = 2p$ and $k = -1$

In both cases, the anticommutation relations of two \overline{Q} 's have to follow the same symmetry pattern as the anticommutation relations (33) of two Q 's. Associativity is realized irrespectively of the values of the parameters α_k . That is to say that we obtained two families of associative algebras \mathcal{A}, \mathcal{B} , each indexed by $\Delta + 1$ parameters and by the integers V and Δ . By a suitable choice of the normalisation of the Q 's and/or of the \overline{Q} 's one can set $\alpha_\Delta = 1$ in (23). One can also set $\alpha_{\Delta-1} = 0$ by using an appropriate translation (7) on the operators $J(N, \Delta)$. Before presenting the proof of this result, let us discuss a few properties of the algebras.

Case 1. The abstract algebra $\mathcal{A}(V, \Delta, \alpha_k)$

In the case $p = 0$, $k = \Delta$, the constraint (33) reads

$$S[A, C]\{Q_{[a_1, a_2 \dots a_\Delta]}, Q_{[c_1, c_2 \dots c_\Delta]}\} = 0 \quad (34)$$

and the same relation has to be imposed on the operators \overline{Q} . Using some combinatoric, one can show that (34) encodes a total number of $C_V^{2\Delta+V}$ independent relations among the anticommutators of two Q 's. Remembering

that there are $C_V^{V+\Delta}$ operators Q , we see easily that the number of constraints is lower than the number of independent anticommutators, that is to say that not all anticommutators are constrained.

The operators (12),(13),(14) constitute a particular representation of the algebras of this type : the ones corresponding to the values α_k determined by (24). For these operators the conditions (34) are trivially realized.

In the case $\Delta = 1$, the relation (33) just implies that all anticommutators of two operators Q vanish (and similarly for two \overline{Q}). The algebra $\mathcal{A}(V, 1)$ is linear, it coincides with the Lie superalgebra denoted $spl(V+1, 1)$ in the classification [12]. If $V = 1$ one recovers the algebra $osp(2, 2)$ (remember the equivalence of $osp(2, 2)$ with $spl(2, 1)$).

Case 2. The abstract algebra $\mathcal{B}(V, \Delta, \alpha_k)$

If Δ is even and if $p = \Delta/2$ the constraints (33) on the Q 's can be set in the form

$$\{Q_{a_1 a_2 \dots a_\Delta}, Q_{a_{\Delta+1} a_{\Delta+2} \dots a_{2\Delta}}\} = \{Q_{\sigma(a_1) \sigma(a_2) \dots \sigma(a_\Delta)}, Q_{\sigma(a_{\Delta+1}) \sigma(a_{\Delta+2}) \dots \sigma(a_{2\Delta})}\} \quad (35)$$

for any permutation σ of the 2Δ indices. The total number of independent constraints is not as easy to find as in the case 1; we obtained it in two particular cases

$$\frac{\Delta(\Delta-1)}{2} \quad \text{if } V = 1 \quad (36)$$

and

$$\frac{V(V+1)(V^2+9V-4)}{12} \quad \text{if } \Delta = 2. \quad (37)$$

Exceptional solutions

It should be stressed that associative algebra could also exist with the same structure as above, i.e. with (5), (17), (18), (33) and (23) but where some of

the parameters α_k are 2×2 diagonal matrices. That is to say they depend on the Casimir operators constructed with the $gl(V+1)$ subalgebra generated by the operators (12). We could not solve this problem for general values of Δ but we studied completely the cases $\Delta = 1, 2, 3$. We obtained one new solution in the case $\Delta = 2, V = 1, p = 1$. The most general relation for $\{Q, \overline{Q}\}$ which is compatible with associativity depends on four parameters. It is of the form

$$\{Q_{a_1 a_2}, \overline{Q}^{b_1 b_2}\} = \sum_{j=0}^2 \alpha_j W_{a_1 a_2}^{b_1 b_2}(j) + \beta \left(C_1 W_{a_1 a_2}^{b_1 b_2}(1) + (4C_2 - 3C_1^2) W_{a_1 a_2}^{b_1 b_2}(0) \right) \quad (38)$$

where β is the additional parameter while C_1, C_2 represent the Casimir operators (6) computed in the representation (12).

5 Proof of proposition 2

Let us come to the proof of proposition 2. The relevant Jacobi identities are

$$\left[\{Q_{[A]}, \overline{Q}^{[B]}\}, Q_{[C]} \right] + \left[\{Q_{[C]}, \overline{Q}^{[B]}\}, Q_{[A]} \right] + \left[\{Q_{[A]}, Q_{[C]}\}, \overline{Q}^{[B]} \right] = 0 . \quad (39)$$

The application of S_Y (see (31)) to this equation and the use of (33) lead to the necessary and sufficient conditions

$$S_Y \left(\left[\{Q_{[A]}, \overline{Q}^{[B]}\}, Q_{[C]} \right] + \left[\{Q_{[C]}, \overline{Q}^{[B]}\}, Q_{[A]} \right] \right) = 0 . \quad (40)$$

Moreover S_Y (with an even second line) applied to a tensor $T_{[A,C]}$, symmetrical in $[A]$ on one side and in $[C]$ on the other side, selects automatically the piece in T symmetrical under the exchange $[A] \leftrightarrow [C]$. Hence, the necessary and sufficient condition becomes simply

$$S_Y \left(\left[\{Q_{[C]}, \overline{Q}^{[B]}\}, Q_{[A]} \right] \right) = 0 . \quad (41)$$

Let us first suppose that the anticommutation relations of Q and \overline{Q} take the form

$$\{Q_{[A]}, \overline{Q}^{[B]}\} = S[A]S[B]J_{a_1}^{b_1}J_{a_2}^{b_2} \dots J_{a_\Delta}^{b_\Delta} \quad (42)$$

rather than the more general one (23). Using (42) together with (17), and separating the terms, say X' , which come out without k (through (17)) from the terms, say kY' , which come out linear in k , the expression (41) becomes

$$X' + kY' = 0 \quad (43)$$

where

$$X' = -S_Y S[B] \sum_{i=1}^{\Delta} \sum_{j=1}^{\Delta} \delta_{a_i}^{b_1} Q_{[c_j, \hat{A}_i]} S[\hat{C}_j] J_{c_1}^{b_2} J_{c_2}^{b_3} \dots J_{c_{\Delta}}^{b_{\Delta}} - \dots \quad (44)$$

$$Y' = S_Y S[B] \sum_{j=1}^{\Delta} \delta_{c_j}^{b_1} Q_{[A]} S[\hat{C}_j] J_{c_1}^{b_2} J_{c_2}^{b_3} \dots J_{c_{\Delta}}^{b_{\Delta}} + \dots \quad (45)$$

In (44) and (45), the \dots refer to the terms where the Q does not appear as the first operator, but rather after a J operator. Remark also that the index c_j is absent in the set $[\hat{C}_j]$ and accordingly does not appear as a lower index in the J 's.. It follows that, as it should, the number $(\Delta - 1)$ of indices b_k in the product of the J 's matches the number of c_m indices.

Since $X' + kY'$ has to be zero identically, every coefficient of every (independent) operator entering in it has to be zero. This allows a great simplification in the necessary and sufficient conditions.

- The terms labeled \dots in (44) and (45) can be forgotten altogether. Indeed, the terms where the Q 's are in the first position are independent of the terms where they are not.
- The symmetry on the $[B]$ can also be eliminated. Every term, for every value of the indices b_k , has to vanish on its own.
- Let us introduce the notations

$$W(a_i) = \delta_{a_i}^{b_1} \quad (46)$$

for some arbitrary fixed value of b_1 and

$$V[\hat{C}_k] = J_{c_1}^{b_2} J_{c_2}^{b_3} \dots J_{c_{\Delta}}^{b_{\Delta}} \quad (47)$$

where c_k is absent as a lower index and b_2, \dots, b_Δ have also fixed values..

With these simplifications, the condition $X' + kY' = 0$ reduces to the necessary and sufficient condition $X + kY = 0$ with

$$X = -S_Y \sum_{i=1}^{\Delta} \sum_{j=1}^{\Delta} W(a_i) Q_{[c_j, \hat{A}_i]} S[\hat{C}_j] V[\hat{C}_j] \quad (48)$$

and

$$Y = S_Y \sum_{j=1}^{\Delta} W(c_j) Q_{[A]} S[\hat{C}_j] V[\hat{C}_j] . \quad (49)$$

The operator $X + kY$ is composed of exactly two types of independent operators. They can be written canonically as

$$O_1 = W(c_1) Q_{[A]} V[\hat{C}_1] , \quad (50)$$

$$O_2 = W(c_\Delta) Q_{[A]} V[\hat{C}_\Delta] . \quad (51)$$

Indeed :

- The indices of the Q operator have to be completely symmetrical. Hence they must belong to the first line of the Young tableau and by symmetry of S_Y can be chosen as the $[A]$ set.
- If the index in W is taken in the first line, it can be chosen to be c_Δ . This is due to the fact that any of the indices (except the those belonging to the set $[A]$ which already pertain to the Q) in the first line is equivalent by symmetry to any other in the first line. The remaining indices for the V can be chosen in any order and for example in the natural order.
- If the index in W is taken in the second line, it can be chosen to be c_1 . Indeed any of the indices in the second line is equivalent by symmetry to any other in the second line. The remaining indices for the V can again be chosen in any order and for example in the natural order.

The remaining task is to extract in X and in Y the number of times the operators O_1 and O_2 occur. This is a rather delicate operation in terms of the symmetries involved. Let us call X_i (resp. Y_i) with $i = 1, 2$ the coefficient of the operator O_i in X (resp. Y). With these notations the condition $X + kY = 0$ becomes equivalent to

$$X_1 + kY_1 = 0 \quad , \quad X_2 + kY_2 = 0 \quad . \quad (52)$$

To now compute these four coefficients, we will make use of the fundamental theorem of group theory which states that, if P is any permutation of the elements in $[A]$

$$S[A] = PS[A] = S[A]P \quad (53)$$

Computation of X_1

Let us rewrite X as

$$X = - \sum_{i=1}^{\Delta} \sum_{j=1}^{\Delta} S[A, c_{2p+1}, \dots, c_{\Delta}] S[c_1, \dots, c_{2p}] E_x W(a_i) Q_{[c_j, \hat{A}_i]} S[\hat{C}_j] V[\hat{C}_j] \quad (54)$$

where we have interchanged the finite summation on i and j with the symmetry operations. First, the a_i in $W(a_i)$ which belongs to the first line has to be replaced by a c belonging to the second line. This can be done at the intervention of the operator E_x only. At the same time none of the other $a_j, j \neq i$ in Q should be replaced by an element of the second line. Hence, from the 2^{2p} terms in E_x we can restrict ourselves to the transposition (a_i, c_i) which comes with a minus sign. At the same time i can be restricted to the range $1, 2p$. The summation on j then has one term with $j = i$. For the terms with $j \neq i$, the c_j in Q has to belong to the set $j = 2p + 1, \dots, \Delta$ in order to be able to replace it by an a by the first symmetry operator S in (54). Hence the restricted part of X , say \hat{X} , is

composed of two pieces, say \hat{X}_α and \hat{X}_β ,

$$\begin{aligned}
\hat{X}_\alpha &= \sum_{i=1}^{2p} S[A, c_{2p+1}, \dots, c_\Delta] S[c_1, \dots, c_{2p}] (a_i, c_i) W(a_i) Q_{[c_i, \hat{A}_i]} S[\hat{C}_i] V[\hat{C}_i] \\
&= \sum_{i=1}^{2p} S[A, c_{2p+1}, \dots, c_\Delta] S[c_1, \dots, c_{2p}] W(c_i) Q_{[A]} S[\hat{C}_i] V[\hat{C}_i] \\
&= \sum_{i=1}^{2p} S[A, c_{2p+1}, \dots, c_\Delta] S[c_1, \dots, c_{2p}] (c_1, c_i) W(c_i) Q_{[A]} S[\hat{C}_i] V[\hat{C}_i] \\
&= S[A, c_{2p+1}, \dots, c_\Delta] S[c_1, \dots, c_{2p}] \sum_{i=1}^{2p} W(c_1) Q_{[A]} S[\hat{C}_1] V[\hat{C}_1] \quad (55)
\end{aligned}$$

and

$$\begin{aligned}
\hat{X}_\beta &= \sum_{i=1}^{2p} \sum_{j=2p+1}^{\Delta} S[A, c_{2p+1}, \dots, c_\Delta] S[c_1, \dots, c_{2p}] (a_i, c_i) \\
&\quad W(a_i) Q_{[c_j, \hat{A}_i]} S[\hat{C}_j] V[\hat{C}_j] . \quad (56)
\end{aligned}$$

Using (53), we easily conclude that in X_α the following coefficient appears

$$(\Delta)!(2p)!(\Delta - 2p)! . \quad (57)$$

The first factor $(\Delta)!$ comes from the permutation of the $[A]$ set which always contributes to an equal factor due to the symmetry of the Q . The second term $(2p)!$ comes from the product of the sommation over i (a factor $2p$) and of a factor $(2p - 1)!$ coming from the repetition of the symmetries in $[c_2, \dots, c_{2p}]$ contained in the second and in the third S factors. The last term $(\Delta - 2p)!$ comes from the repetition of the symmetries in $[c_{2p+1}, \dots, c_\Delta]$ contained in the first and in the third S factors.

Let us now focuss our attention on the X_β term. Using (53) we can factor out of $S[A, c_{2p+1}, \dots, c_\Delta]$, at no cost, a transposition factor (a_i, c_j) and from $S[c_1, \dots, c_{2p}]$ a factor (c_1, c_i) . The product of these two transpositions together with the transposition in X_β leads to the cyclic permutation (a_i, c_1, c_i, c_j) and the relevant part \hat{X}_β becomes

$$\hat{X}_\beta = \sum_{i=1}^{2p} \sum_{j=2p+1}^{\Delta} S[A, c_{2p+1}, \dots, c_\Delta] S[c_1, \dots, c_{2p}]$$

$$W(c_1)Q_{[A]}S[\hat{C}_1]V[\hat{C}_1] . \quad (58)$$

The following coefficient then appears

$$(\Delta - 2p)(\Delta)!(2p)!(\Delta - 2p)! . \quad (59)$$

The extra factor as compared to the coefficient coming out of X_α is due to the extra sommation over j .

Summing up the results (59,57), we find

$$X_1 = (1 + \Delta - 2p)(\Delta)!(2p)!(\Delta - 2p)! . \quad (60)$$

Computation of Y_1

The same technique applied to Y_1 is much simpler as the relevant term in E_x is simply the identity. Hence

$$\begin{aligned} \hat{Y} &= S[A, c_{2p+1}, \dots, c_\Delta]S[c_1, \dots, c_{2p}] \sum_{j=1}^{2p} W(c_j)Q_{[A]}S[\hat{C}_j]V[\hat{C}_j] \\ &= S[A, c_{2p+1}, \dots, c_\Delta]S[c_1, \dots, c_{2p}] \sum_{j=1}^{2p} W(c_1)Q_{[A]}S[\hat{C}_1]V[\hat{C}_1] . \end{aligned} \quad (61)$$

To pass from the first to the second line we have factored out of $S[c_1, \dots, c_{2p}]$ the transposition (c_1, c_j) .

Collecting again the factors, we find

$$Y_1 = (\Delta)!(2p)!(\Delta - 2p)! . \quad (62)$$

Computation of X_2

The relevant term in E_x is again the identity and the reduced part of X which can lead to a term of the form O_2 (51) is

$$\hat{X} = - \sum_{i=1}^{\Delta} \sum_{j=1}^{\Delta} S[A, c_{2p+1}, \dots, c_\Delta]S[c_1, \dots, c_{2p}]W(a_i)Q_{[c_j, \hat{A}_i]}S[\hat{C}_j]V[\hat{C}_j] . \quad (63)$$

The sommation on j has to be restricted to those values in the first line of the Young diagram. A transposition (a_i, c_j) can then be factored out of $S[A, c_{2p+1}, \dots, c_\Delta]$ as well as a transposition (a_i, c_Δ) , i.e. in total a cyclic permutation (a_i, c_Δ, c_j) . We find

$$\begin{aligned}
\hat{X} &= - \sum_{i=1}^{\Delta} \sum_{j=2p+1}^{\Delta} S[A, c_{2p+1}, \dots, c_\Delta] S[c_1, \dots, c_{2p}] (a_i, c_\Delta, c_j) \\
&\quad W(a_i) Q_{[c_j, \hat{A}_i]} S[\hat{C}_j] V[\hat{C}_j] \\
&= - \sum_{i=1}^{\Delta} S[A, c_{2p+1}, \dots, c_\Delta] S[c_1, \dots, c_{2p}] \sum_{j=2p+1}^{\Delta} \\
&\quad W(c_\Delta) Q_{[A]} S[\hat{C}_\Delta] V[\hat{C}_\Delta] .
\end{aligned} \tag{64}$$

Collecting the factors as usual, we find

$$X_2 = -\Delta(\Delta)!(2p)!(\Delta - 2p)! . \tag{65}$$

The extra factor (Δ) comes from the sommation over i .

Computation of Y_2

In this last case the relevant term in E_x is again the identity and the reduced part of Y which can lead to a term of the form O_2 (51) is

$$\begin{aligned}
\hat{Y} &= S[A, c_{2p+1}, \dots, c_\Delta] S[c_1, \dots, c_{2p}] \sum_{j=2p+1}^{\Delta} W(c_j) Q_{[A]} S[\hat{C}_j] V[\hat{C}_j] \\
&= S[A, c_{2p+1}, \dots, c_\Delta] S[c_1, \dots, c_{2p}] \sum_{j=2p+1}^{\Delta} (c_j, c_\Delta) W(c_j) Q_{[A]} S[\hat{C}_j] V[\hat{C}_j] \\
&= S[A, c_{2p+1}, \dots, c_\Delta] S[c_1, \dots, c_{2p}] \sum_{j=2p+1}^{\Delta} W(c_\Delta) Q_{[A]} S[\hat{C}_\Delta] V[\hat{C}_\Delta] .
\end{aligned} \tag{66}$$

Collecting the terms, we find

$$Y_2 = (\Delta)!(2p)!(\Delta - 2p)! . \tag{67}$$

We can now summarize the conditions coming from (52) :

1. The conditions coming from the Jacobi identities are thus two in number if $p \neq 0$ (the condition for the operator O_1 to be defined) and if $\Delta \neq 2p$ (the condition for 0_2 to be defined). These conditions

$$k = -(\Delta + 1 - 2p) \quad (68)$$

$$k = \Delta \quad (69)$$

are incompatible.

2. More generally, the anticommutators of two Q 's cannot vanish for more than one representation.
3. If $p = 0$ the only condition comes from the 0_2 operator. It is

$$k = \Delta \quad (70)$$

which is a solution to our problem. The corresponding Young diagram has only one line of length 2Δ .

4. If Δ is even and $\Delta = 2p$ the only condition comes from the 0_1 operator. It is

$$k = -1 \quad (71)$$

which is a second solution to our problem. The corresponding Young diagram has two lines of equal length Δ .

This achieves the proof of the proposition when the anticommutator $\{Q, \overline{Q}\}$ is restricted to (42). It is easy to see that the other allowed terms in the anticommutator of the Q 's with the \overline{Q} 's, i.e. those which do not involve J 's only but the products of J 's and δ 's as in (23) lead to exactly the same restrictions. Hence they can all be present at the same time leaving us with the form (23) with the $\Delta + 1$ arbitrary coefficients.

The conditions coming from the Jacobi identities involving two \overline{Q} and one Q also lead to exactly the same conditions. Hence the anticommutators which are chosen to be zero for the anticommutations of two Q 's on one side or of two \overline{Q} 's on the other side must be identical.

6 Summary and Conclusions

The operators preserving globally a system of two polynomials in V variables ($V \geq 1$) and of degrees N and $N - \Delta$ ($\Delta \geq 0$) respectively can be constructed as the elements of the enveloping algebra of certain superalgebras.

In this paper, we have constructed a family of such associative, non linear superalgebras. Any of these algebras is specified by V , by Δ and by a set of $\Delta + 1$ complex numbers noted α_k with $k = 0, 1, \dots, \Delta$. They are generated by $(V + 1)^2$ (bosonic) operators

$$J_a^b \quad , \quad a, b = 0, 1, \dots, \Delta \quad (72)$$

and by two sets of $C_\Delta^{V+\Delta}$ (fermionic) operators

$$Q_{[a_1, \dots, a_\Delta]} \quad , \quad \overline{Q}^{[a_1, \dots, a_\Delta]} \quad , \quad a_k = 0, 1, \dots, \Delta \quad (73)$$

symmetric in their Δ indices.

The bosonic generators obey the commutation relations of the Lie algebra $gl(V + 1)$. The operators Q (and separately the \overline{Q}) assemble into a specific representation of $gl(V + 1)$ under the adjoint action of J_a^b (see (17),(18)). The anticommutators $\{Q, \overline{Q}\}$ are polynomials of degree at most Δ in the bosonic operators. The arbitrariness of the polynomials is coded in the $\Delta + 1$ parameters α_k (23).

All the supplementary conditions on the products of the operators Q (and of the operators \overline{Q}) necessary to guarantee associativity (equivalent to the generalised Jacobi identities) are given by our proposition 2.

For all fixed values of the integers V and Δ and of the complex parameters α_k we denote $\mathcal{A}(V, \Delta, \alpha_k)$ the algebra corresponding to case 1 of proposition 2. If Δ is even, a supplementary algebraic structure, that we denote $\mathcal{B}(V, \Delta, \alpha_k)$ is also possible, as predicted by case 2 of proposition 2.

Referring to the general definition of a W -algebra given recently in [13], it is natural to classify $\mathcal{A}(V, \Delta, \alpha_k)$ and $\mathcal{B}(V, \Delta, \alpha_k)$ as “finite $W_{\Delta+1}$ -superalgebras”.

An analysis of the representations of $\mathcal{A}(1, 2, \alpha_0, \alpha_1, \alpha_2)$, performed recently [11], leads to a rather rich set of inequivalent irreducible, finite dimensional representations .

Let us stress again that the operators in the enveloping algebras that we have constructed are directly relevant for the study of quasi-exactly solvable systems of equations.

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